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1987 J. Phys. A: Math. Gen. 20 5719

(http://iopscience.iop.org/0305-4470/20/16/044)

J. Phys. A: Math. Gen. **20** (1987) 5719-5721. Printed in the UK

COMMENT

On integrals involving associated Legendre functions and powers of $(1-x^2)$

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Received 3 March 1987, in final form 13 April 1987

Abstract. The integrals involving associated Legendre functions and the factor $(1-x^2)^{-p-1}$, $p>0$, have been extended for negative values of p using a transformation of a terminating type of hypergeometric series.

Recently (Laursen and Mita 1981, Ullah 1982, Rashid 1986) there has been much interest in the evaluation of integrals involving two associated Legendre functions and powers of $(1-x^2)$ in terms of a terminating generalised hypergeometric series. These integrals occur in meson physics (Laursen and Mita 1981) and can be used to derive new sum rules for the vector coupling coefficients (Ullah 1982). In the latest study of these integrals (Rashid 1986) it is pointed out that the final form of the integral (Ullah 1984) involving two associated Legendre functions and the factor $(1-x^2)^{-p-1}$ does not remain of terminating type for positive powers of $(1-x^2)$ or negative values of p. The purpose of the present comment is to extend the expressions for the integrals given earlier (Ullah 1984) so that they are valid both for positive and negative powers of the factor $(1-x^2)^{-p-1}$. We show below that this can be done by using a transformation on the generalised hypergeometric series given by Bailey (1935).

Let us first consider the integral

$$I_p=\int_{-1}^{1}\mathrm{d}x\frac{[P_l^m(x)]^2}{(1-x^2)^{p+1}} \tag{1}$$

where P_l^m is an associated Legendre function and l, m, p are positive integers with $l \geqslant m$, $p \leqslant (2m-1)/2$.

The operator form of Taylor's theorem gives the following expression (Ullah 1984) for I_p:

$$I_p=\frac{(l+m)!(m-p-1)!\Gamma(\frac{1}{2}(l+m+1))\Gamma(\frac{1}{2}(l-m+2)+p)}{(l-m)!m!p!\Gamma(\frac{1}{2}(l+m+1)-p)\Gamma(\frac{1}{2}(l-m+2))}$$
$$\times{}_4F_3(\tfrac{1}{2}(l+m+1),-\tfrac{1}{2}(l-m),m-p,-p;$$
$$m+1,\tfrac{1}{2}(l+m+1-2p),\tfrac{1}{2}(-l+m-2p);1). \tag{2}$$

It is obvious that ${}_4F_3$ in expression (3) is of terminating type as long as $p>0$.

We would now like to apply a transformation on it so that it is of terminating type both for negative and positive values of p.

It is shown by Bailey (1935) that a terminating-type hypergeometric series can be transformed as

$$ {}_4F_3\left[\begin{matrix} x, y, z, -n; \\ u, v, w \end{matrix}\right] = \frac{(v-z)_n(w-z)_n}{(v)_n(w)_n}\, {}_4F_3\left[\begin{matrix} u-x, u-y, z, -n; \\ 1+v+z-n, 1-w+z-n, u \end{matrix}\right] \tag{3} $$

provided $u+v+w=x+y+z-n+1$.

In expression (3), the symbol $(z)_n$ stands for $\Gamma(z+n)/\Gamma(z)$, and a slightly different notation for hypergeometric series with argument 1 is used.

In the transformation (3) we put

$$ \begin{array}{llll} x=m-p & y=\frac{1}{2}(l+m+1) & z=-\frac{1}{2}(l-m) & n=p \\ u=m+1 & v=\frac{1}{2}(l+m+1-2p) & w=\frac{1}{2}(-l+m-2p). & \end{array} \tag{4} $$

As can be checked these parameters satisfy the condition $u+v+w=x+y+z-n+1$.

Using (2)-(4) and some algebraic steps we find that I_p can also be written as

$$ I_p = \frac{(l+m)!(m-p-1)!\Gamma(\frac{1}{2}(2l+1))}{(l-m)!m!\Gamma(\frac{1}{2}(2l+1-2p))} \times {}_4F_3\left[\begin{matrix} p+1, \frac{1}{2}(-l+m+1), \frac{1}{2}(-l+m), -p; \\ \frac{1}{2}(-2l+1), 1, m+1 \end{matrix}\right]. \tag{5} $$

Expression (5) gives I_p in terms of a hypergeometric series which is terminating both for positive and negative values of p.

As an application of expression (5) we derive the integral for $p=-2$. Using the notation (Rashid 1986) $J(l, l, m; -2)$ for this integral, we find from expression (5) that it is given by

$$ J(l, l, m; -2) = 4\frac{l(l+1)+(m-1)(m+1)}{(2l-1)(2l+1)(2l+3)}\frac{(l+m)!}{(l-m)!}. \tag{6} $$

Comparing with the tabulated value (Rashid 1986) we find that a factor of four is missing in the tabulated value. One could further check that the result given by expression (6) which is obtained from expression (5) is correct while the tabulated one is not by calculating several simple integrals involving $[P_l^m(x)]^2$ and $(1-x^2)$. All of them give the same value as one obtains from expression (6).

If one wishes, one could also recast the more general integral involving two associated Legendre functions (Ullah 1984) using transformation (3). This integral is written as (Ullah 1984)

$$ I_p = \int_{-1}^{1} \mathrm{d}x \frac{P_l^m(x)P_k^n(x)}{(1-x^2)^{p+1}} \tag{7} $$

where l, m, k, n, p are positive integers, $l \geqslant m$, $k \geqslant n$, $p \leqslant (m+n-1)/2$ and $l+k-(m+n)$ even.

This integral can be rewritten as

$$ I_p = 2^{m-n} \frac{\Gamma(\frac{1}{2}(l+k)+\frac{1}{2}(m-n)+\frac{1}{2})\Gamma(\frac{1}{2}(m+n)-p)\Gamma(k+n+1)\Gamma(p-\frac{1}{2}k+\frac{1}{2}l+1)}{\Gamma(k-n+1)\Gamma(n+1)\Gamma(\frac{1}{2}(l+k)-p+\frac{1}{2})\Gamma(p-\frac{1}{2}n+\frac{1}{2}m+1)\Gamma(\frac{1}{2}l-\frac{1}{2}m-\frac{1}{2}k+\frac{1}{2}n+1)} \times {}_4F_3\left[\begin{matrix} p+1-\frac{1}{2}(m-n), -\frac{1}{2}k+\frac{1}{2}n+\frac{1}{2}, -\frac{1}{2}k+\frac{1}{2}n, -p-\frac{1}{2}(m-n); \\ -\frac{1}{2}k-\frac{1}{2}l+\frac{1}{2}(n-m)+\frac{1}{2}, -\frac{1}{2}k+\frac{1}{2}l+\frac{1}{2}(n-m)+1, n+1 \end{matrix}\right]. \tag{8} $$

As expected if we put $k=l$ and $n=m$, expression (8) becomes the same as the earlier expression given by (5).

Lastly we remark that, since for positive powers $(1-x^2)^r$ can be written as an associated Legendre polynomial $P_{2r}^{2r}(x)$ apart from a known constant, the integral of two associated Legendre polynomials and positive powers of $(1-x^2)$ can be written in terms of vector coupling coefficients (Ullah 1982) for the case where the magnetic quantum number is conserved, and therefore its numerical value can be obtained from a table of vector coupling coefficients.

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